B.sc. part - II (H), Differential calculus
paper - III (H) Successive Differentiation.

Problems based on the nth derivative.

Q.1. If $y = \sin mx + \cos mx$, prove that
$y_n = m^n [1 + (-1)^n \sin 2mx]^{\frac{1}{2}}$.

Soln: We have $y = \sin mx + \cos mx$

$\therefore y_n = m^n \sin\left(mx + \frac{n\pi}{2}\right) + m^n \cos\left(mx + \frac{n\pi}{2}\right)$

$\Rightarrow y_n = m^n \left[\left\{\sin\left(mx + \frac{n\pi}{2}\right) + \cos\left(mx + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$

$\Rightarrow y_n = m^n \left[\sin^2\left(mx + \frac{n\pi}{2}\right) + \cos^2\left(mx + \frac{n\pi}{2}\right) + 2\sin\left(mx + \frac{n\pi}{2}\right)\cos\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$

$= m^n \left[1 + \sin 2\left(mx + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$

$= m^n \left[1 + \sin(n\pi + 2mx)\right]^{\frac{1}{2}}$

$= m^n \left[1 \pm \sin 2mx\right]^{\frac{1}{2}} = m^n \left[1 + (-1)^n \sin 2mx\right]^{\frac{1}{2}}$

[$\because$ as n is even or odd]

Q. 2. If $y = \frac{1}{x^2 + a^2}$, find y_n

Soln. We have $y = \frac{1}{x^2 + a^2} = \frac{1}{x^2 - (-1)a^2}$

$\Rightarrow y = \frac{1}{x^2 - i^2 a^2} = \frac{1}{(x - ia)(x + ia)}$

$= \frac{1}{2ia}\left[\frac{(x + ia) - (x - ia)}{(x - ia)(x + ia)}\right]$

$= \frac{1}{2ia}\left[\frac{x + ia}{(x - ia)(x + ia)} - \frac{x - ia}{(x - ia)(x + ia)}\right]$

$= \frac{1}{2ia}\left[\frac{1}{x - ia} - \frac{1}{x + ia}\right]$

B.Sc Part-II (H) Diff. Calculus 13-07-2020

1 Paper-III Successive Differentiation

Rest part of Q. 2.

Differentiating both sides n times w.r.t. x, we get

$$y_n = \frac{1}{2ia}\left[\frac{(-1)^n \lfloor n}{(x-ia)^{n+1}} - \frac{(-1)^n \lfloor n}{(x+ia)^{n+1}}\right]$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right]$$

Put $x = r\cos\theta$ and $a = r\sin\theta$.

Squaring and adding these two, we get

$$x^2+a^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2 \cdot 1 = r^2$$

and by division, we get

$$\frac{x}{a} = \frac{r\cos\theta}{r\sin\theta} = \cot\theta \quad \therefore \theta = \cot^{-1}\frac{x}{a}.$$

$$\therefore y_n = \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia}\left[\frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}} - \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia \cdot r^{n+1}}\left[(\cos\theta + i\sin\theta)^{-(n+1)} - (\cos\theta - i\sin\theta)^{-(n+1)}\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia\, r^{n+1}}\left[\cancel{\cos(n+1)\theta} + i\sin(n+1)\theta - \cancel{\cos(n+1)\theta} + i\sin(n+1)\theta\right]$$

$$= \frac{(-1)^n \lfloor n}{2ia \cdot r^{n+1}}\, 2i\sin(n+1)\theta$$

$$\Rightarrow y_n = \frac{(-1)^n \lfloor n}{a\left(\frac{a}{\sin\theta}\right)^{n+1}} \sin(n+1)\theta.$$

Hence $y_n = \dfrac{(-1)^n \lfloor n}{a^{n+2}} \sin^{n+1}\theta \, \sin(n+1)\theta.$

where $\theta = \cot^{-1}\dfrac{x}{a}.$